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Group classification of two-dimensional steady viscous gas dynamics equations with arbitrary state equations

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Abstract
This paper is devoted to the group classification of steady viscous gas dynamics equations in the two-dimensional case (with plane or cylindrical symmetry) with arbitrary state equations. Representations of all invariant solutions are given.

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1. Introduction

The analytic study of the properties of partial differential equations plays an important role in applied mathematics and mathematical physics. One of the methods for studying the properties of differential equations is group analysis. The modern state of group analysis is reviewed in [1]. Group analysis besides constructing exact solutions provides a regular procedure for mathematical modelling by classifying differential equations with respect to arbitrary elements. The application of group analysis implies some steps. The first step is a group classification with respect to arbitrary elements. An admitted group is found at this step. The next step is a construction of an optimal system of subalgebras. Then one can attempt to find an invariant or partially invariant solution for each subalgebra of the optimal system.

We should note here that many invariant solutions of the viscous gas dynamics equations have also been obtained by other methods [2–10]. The group classification of the viscous gas dynamics[1] equations was done in [11]. The group classification of two-dimensional steady viscous gas dynamics equations for an ideal gas was done in [12]. For some models of viscous gas dynamics equations, group analysis was used in [13]. Unsteady spherically symmetric viscous gas dynamics equations were studied in [14].

[1] Here the first, $\lambda = \lambda(T)$, and second, $\mu = \mu(T)$, coefficients of viscosity are related by the equation $\lambda = -2\mu/3$, and $\kappa = \kappa(T)$.

This paper is devoted to the application of group analysis for studying the viscous gas dynamics equations with arbitrary state equations.

2. The group analysis algorithm

Let us first review the notations and techniques used in group analysis.

Let an lth-order system of differential equations

$$(S)\colon F^k(x,u,p)=0 \qquad (k=1,2,\ldots,s)$$

be given. Here $x=(x_i)$, $(i=1,2,\ldots,n)$, are the independent variables, $u=(u^j)$ $(j=1,2,\ldots,m)$ are the dependent variables, $p=\left(p^k_\alpha\right)$ are the derivatives up to lth-order and $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_n)$ is a multi-index with $|\alpha|\equiv\alpha_1+\alpha_2+\cdots+\alpha_n\leqslant l$.

2.1. Admitted Lie group of transformations

One of the main objects in group analysis is the local one-parameter Lie group G^1 of the transformations:

$$x_i'=f^{x_i}(x,u;a) \qquad u^{j\prime}=f^{u^j}(x,u;a) \qquad (i=1,2,\ldots,n;\ j=1,2,\ldots,m). \tag{1}$$

There is a one-to-one correspondence between groups G^1 and infinitesimal generators

$$X=\xi^i(x,u)\partial_{x_i}+\zeta^j(x,u)\partial_{u^j}$$

where

$$\xi^i(x,u)=\left(\frac{\mathrm{d}f^{x_i}}{\mathrm{d}a}\right)_{|a=0} \qquad \zeta^j(x,u)=\left(\frac{\mathrm{d}f^{u^j}}{\mathrm{d}a}\right)_{|a=0}.$$

The operator

$$\underset{l}{X}=X+\sum_{j,\alpha}\zeta^j_\alpha\partial_{p^j_\alpha}$$

with coefficients

$$\zeta^j_{\alpha,k}=D_k\zeta^j_\alpha-\sum_i p^j_{\alpha,i}D_k\xi^i. \tag{2}$$

is called the lth prolongation of a generator X. Here

$$D_k=\frac{\partial}{\partial x_k}+\sum_{j,\alpha}p^j_{\alpha,k}\frac{\partial}{\partial p^j_\alpha}$$

are the operators of total differentiation with respect to x_k $(k=1,2,\ldots,n)$.

The algorithm for finding a local one-parameter Lie group (1) admitted by the system of differential equations (S) consists of the following four steps.

In the first step, the form of the generator

$$X=\xi^i(x,u)\partial_{x_i}+\zeta^j(x,u)\partial_{u^j}$$

is given, with unknown coefficients $\xi^i(x,u),\zeta^j(x,u)$. In the second step the prolonged operator $\underset{l}{X}$ is applied to every equation of the system (S). In the next step the coefficients of the prolonged operator are substituted by using formulae (2). The equations obtained must be considered on the manifold (S). As a result one obtains the system of differential equations

$$DS\colon \underset{l}{X}F^k(x,u,p)_{|(S)}=0 \qquad (k=1,2,\ldots,s). \tag{3}$$

This system of equations is called the system of determining equations and is an overdetermined system of linear homogeneous differential equations in the unknown coordinates $\xi^i(x, u), \zeta^j(x, u)$. The general solution of the determining equations DS generates a full group GS of the system (S). The feature of the admitted group is that under the action of any transformation of this group, every solution $u = U(x)$ of the system (S) is transformed into a solution $u = U_a(x)$ of the same system (S). Therefore, the admitted group allows construction of new solutions from known solutions. Note that the set of admitted generators generate a Lie algebra, which is called admitted by the system (S).

2.2. *Equivalence transformations*

Most systems of partially differential equations have arbitrary elements: arbitrary functions or arbitrary constants. These arbitrary elements can be separated into classes with respect to a group of equivalence transformations. An equivalence transformation is a nondegenerate change of dependent and independent variables and arbitrary elements, which transforms any system of differential equations of a given class to a system of equations of the same class. These transformations allow us to use the simplest representation of the given equations. Note that the admitted group depends on specialization of the arbitrary elements. The group classification problem consists in searching for an admitted group of transformations, which is admitted for all arbitrary elements of the system and all specializations of the arbitrary elements. The specialization of the arbitrary elements can extend the admitted group. For the calculation of equivalence transformations, we follow the approach developed in [15, 16], which consists of the following.

Suppose, the system of differential equation

$$F^k(x, u, p, \phi) = 0 \qquad (k = 1, 2, \ldots, s) \tag{4}$$

has arbitrary elements $\phi = (\phi^1, \phi^2, \ldots, \phi^t)$, which are functions (or constants) $\phi = \phi(x, u)$. A specific value of the arbitrary elements represents a concrete system of differential equations.

The problem of finding an equivalent transformation consists of constructing a transformation of the space $R^{n+m+t}(x, u, \phi)$ which preserves the equations by only changing their representative $\phi = \phi(x, u)$. For this purpose, we consider the one-parameter group of transformations of the space R^{n+m+t}:

$$x' = f^x(x, u, \phi; a) \qquad u' = f^u(x, u, \phi; a) \qquad \phi' = f^\phi(x, u, \phi; a). \tag{5}$$

A generator of this group has the form

$$X^e = \xi^x \partial_x + \zeta^u \partial_u + \zeta^\phi \partial_\phi \tag{6}$$

with the coordinates[2]:

$$\xi^i = \xi^i(x, u, \phi) \qquad \zeta^{u^j} = \zeta^{u^j}(x, u, \phi) \qquad \zeta^{\phi^k} = \zeta^{\phi^k}(x, u, \phi)$$
$$(i = 1, \ldots, n;\ j = 1, \ldots, m;\ k = 1, \ldots, t).$$

We use the main feature of the Lie group that any solution $u_0(x)$ of system (4) with functions $\phi(x, u)$ is transformed by (5) into another solution $u = u_a(x')$ of system (4), but with different (transformed) functions $\phi_a(x, u)$, which are defined in the following way. Solving the relations

$$x' = f^x(x, u, \phi(x, u); a) \qquad u' = f^u(x, u, \phi(x, u); a)$$

[2] Later the author discovered that similar assumptions about the coefficients of the operator were used in [17] for one class of ordinary differential equations with one nonessential restriction $\zeta^{\phi^k} = \zeta^{\phi^k}(x, \phi)$.

with respect to (x, u), we obtain

$$x = g^x(x', u'; a) \qquad u = g^u(x', u'; a). \tag{7}$$

Then the transformed function is

$$\phi_a(x', u') = f^\phi(x, u, \phi(x, u); a) \tag{8}$$

where instead of (x, u) we have to substitute their expressions (7). The transformed solution $u_a(x)$ is obtained by solving the relations

$$x' = f^x(x, u_0(x), \phi(x, u_0(x)); a)$$

with respect to (x):

$$x = \psi^x(x'; a)$$

and substituting into

$$u_a(x') = f^u(x, u_0(x), \phi(x, u_0(x)); a). \tag{9}$$

The formulae for the transformations of the partial derivatives p_a and the derivatives of the functions ϕ are obtained by differentiating (8) and (9) with respect to x' and u'.

The method for finding a group of equivalence transformations is similar to the algorithm for finding an admitted group of transformations. The difference only consists of the prolongation of the infinitesimal generator X^e. In agreement with the construction of the functions $u_a(x')$ and $\phi_a(x', u')$, the prolonged operator

$$\bar{X}^e = X^e + \zeta^{u_x}\partial_{u_x} + \zeta^{\phi_x}\partial_{\phi_x} + \zeta^{\phi_u}\partial_{\phi_u} + \cdots$$

has the following coordinates

$$\zeta^{u_\lambda} = D^e_\lambda \zeta^u - u_x D^e_\lambda \xi^x \qquad (\lambda = x_1, x_2, \ldots, x_n)$$

with $D^e_\lambda = \partial_\lambda + u_\lambda\partial_u + (\phi_u u_\lambda + \phi_\lambda)\partial_\phi$ and

$$\zeta^{\phi_\lambda} = \tilde{D}^e_\lambda \zeta^\phi - \phi_x \tilde{D}^e_\lambda \xi^x - \phi_u \tilde{D}^e_\lambda \zeta^u \qquad (\lambda = u^1, u^2, \ldots, u^m, x_1, x_2, \ldots, x_n)$$

with $\tilde{D}^e_\lambda = \partial_\lambda + \phi_\lambda\partial_\phi$.

An equivalence group GS^e of transformations is generated by $G^1(X^e)$.

Remark 1. In some cases one may have additional requirements for the arbitrary elements. For example, the arbitrary elements ϕ^μ may be supposed to be independent of the independent variables $\frac{\partial \phi^\mu}{\partial x_k} = 0$. When studying the equivalence group, such conditions have to be added to the original system of differential equations (4), leading to additional determining equations.

Remark 2. Note that in the case of the Navier–Stokes equations, kinematic viscosity is the arbitrary element and these equations can be transformed to equations (14) by scaling the independent and dependent variables.

2.3. Invariant and partially invariant solutions

For each subgroup of the admitted group GS, one can try to find an invariant or partially invariant solution. Let $H \subset GS$ be a group admitted by the system of equations (S). Assume that $X_1, \ldots, X_r$ is a basis of the Lie algebra L^r which corresponds to the group H. An invariant or partially invariant solution with respect to the group H is called an H-solution. The method [18] for constructing H-solutions with respect to the group H requires us to find a universal invariant of this group: a set of all functionally independent invariants. For this purpose one needs to solve the overdetermined linear system of differential equations:

$$X_i \phi(x, u) = 0 \qquad (i = 1, 2, \ldots, r). \tag{10}$$

Because $X_1, \dots, X_r$ generate a Lie algebra, system (10) is complete. Its general solution can be expressed through the $m+n-r_*$ invariants

$$J = (J^1(x,u), J^2(x,u), \dots, J^{m+n-r_*}(x,u))$$

where r_* is the total rank of the matrix composed of the coefficients of the generators $X_i\ (i = 1, 2, \dots, r)$. If the rank of the Jacobi matrix $\frac{\partial(J^1,\dots,J^{m+n-r_*})}{\partial(u_1,\dots,u_m)}$ is equal to q, then without loss of generality, one can choose the first $q \leqslant m$ invariants $J^1, \dots, J^q$ such that the rank of the Jacobi matrix $\frac{\partial(J^1,\dots,J^q)}{\partial(u_1,\dots,u_m)}$ is equal to q and the remaining $k = m+n-r_*-q$ invariants $J^{q+1}, J^{q+2}, \dots, J^{m+n-r_*}$ only depend on the independent variables x. H-solutions are characterized by two integers: the rank $\sigma = \delta + n - r_* \geqslant 0$ and the defect $\delta \geqslant 0$; thus one uses the notation $H(\sigma, \delta)$-solution. The rank and defect must satisfy the inequalities

$$k \leqslant \sigma < n \qquad \max\{r_* - n, m - q, 0\} \leqslant \delta \leqslant \min\{r_* - 1, m - 1\}.$$

To construct a representation of $H(\sigma, \delta)$-solutions, one needs to separate the universal invariant into two parts: $J = (\bar{J}, \bar{\bar{J}})$, where $l = m - \delta$ and

$$\bar{J} = (J^1, \dots, J^l) \qquad \bar{\bar{J}} = (J^{l+1}, J^{l+2}, \dots, J^{m+n-r_*}).$$

This means that one can choose the number l such that $1 \leqslant l \leqslant q \leqslant m$. The rank and defect of the $H(\sigma, \delta)$-solution are $\delta = m - l, \sigma = m + n - r_* - l = \delta + n - r_*$. A solution is called invariant if $\delta = 0$, otherwise it is called a partially invariant solution. From the first l invariants $J^1, J^2, \dots, J^l$ one can define the l dependent functions

$$u^i = \phi^i(\bar{J}, u^{l+1}, u^{l+2}, \dots, u^m, x) \qquad (i = 1, \dots, l). \tag{11}$$

The functions $u^{l+1}, u^{l+2}, \dots, u^m$ are called superfluous. The representation of the $H(\sigma, \delta)$-solution is obtained by assuming that the first part of the universal invariant is a function of the second part:

$$\bar{J} = \Psi(\bar{\bar{J}}) \tag{12}$$

and substituting (12) into (11). Thus, the representation of an invariant or partially invariant solution is

$$u^i = \Phi^i(\bar{\bar{J}}, u^{l+1}, u^{l+2}, \dots, u^m, x) \qquad (i = 1, \dots, l) \tag{13}$$

where $\Phi^i = \phi^i(\Psi(\bar{\bar{J}}), u^{l+1}, u^{l+2}, \dots, u^m, x)$.

If $\delta \neq 0$, then either $\sigma = k$ or $\sigma > k$. In the first case ($\sigma = k$) the partially invariant solution is called regular, otherwise it is called irregular [19]. The number $\sigma - k$ is called the measure of irregularity.

After constructing the representation of an invariant or partially invariant solution one needs to substitute it into the original system of equations. The system of equations in the functions Ψ^i and the superfluous functions thus obtained is called the reduced system. This system is overdetermined and requires analysis of compatibility. Usually the compatibility analysis is easier for invariant solutions than for the partially invariant ones.

If H' is a subgroup of H, then it may be possible that a partially invariant $H(\sigma, \delta)$-solution is a partially invariant $H'(\sigma', \delta')$-solution. In this case $\delta' \leqslant \delta, \sigma' \geqslant \sigma$ [18]. A solution is called reducible to a $H'(\sigma', \delta')$-solution if there exists $H' \subset H$ such that $\delta' < \delta, \sigma' = \sigma$. In particular, a solution is called reducible to an invariant solution if there exists $H' \subset H$ with $\delta' = 0$. Thus, a natural problem is to reduce a partially invariant $H(\sigma, \delta)$-solution to an invariant $H'(\sigma, 0)$-solution.

3. Viscous gas dynamics equations

The viscous gas dynamics equations govern the three-dimensional motion of a compressible, thermal conductive, Newtonian viscous gas flow

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \tau\,\mathrm{div}(P) \qquad \frac{\mathrm{d}\tau}{\mathrm{d}t} - \tau\,\mathrm{div}(\boldsymbol{v}) = 0 \qquad \frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = \tau P{:}D + \tau\,\mathrm{div}(\kappa\nabla T).$$

Here $\tau = 1/\rho$ is the specific volume, ρ is the density, $\boldsymbol{v}$ is the velocity, P is the stress tensor, $D = \frac{1}{2}\left(\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}} + \left(\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}}\right)^*\right)$ is the rate-of-strain tensor, ε is the internal energy, T is the temperature and κ is the coefficient of heat conductivity. The Stokes axioms for a viscous gas give

$$P = (-p + \lambda\,\mathrm{div}(v))I + 2\mu D$$

where p is the pressure, λ and μ are the first and second coefficients of viscosity, respectively. These coefficients of viscosity are related to the coefficient of bulk viscosity k by the expression

$$k = \lambda + \tfrac{2}{3}\mu.$$

In general, it is believed that k is negligible except in the study of the structure of shock waves and in the absorption and attenuation of acoustic waves.

A viscous gas is a two-parametric medium. As the main thermodynamic variables, we choose pressure p and specific volume τ: entropy η, internal energy ε and temperature T are functions of pressure and specific volume

$$\eta = \eta(p,\tau) \qquad \varepsilon = \varepsilon(p,\tau) \qquad T = T(p,\tau).$$

The first and second thermodynamic laws require these functions to satisfy the equations

$$\eta_p = \frac{\varepsilon_p}{T} \qquad \eta_\tau = \frac{\varepsilon_\tau + p}{T} \qquad 3\lambda + 2\mu \geqslant 0 \quad \mu \geqslant 0 \quad \kappa \geqslant 0.$$

For simplicity of classification we study the case which corresponds to an essentially viscous and heat conductive gas

$$\mu \neq 0 \qquad \kappa \neq 0.$$

Thus, the viscous gas dynamics equations we study are

$$\begin{aligned}
&\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} + \tau\nabla p = \tau((\lambda+\mu)\nabla(\mathrm{div}(\boldsymbol{v})) + (\mathrm{div}(\boldsymbol{v}))\nabla\lambda + \mu\triangle\boldsymbol{v} + 2D(\nabla\mu))\\
&\frac{\mathrm{d}\tau}{\mathrm{d}t} - \tau\,\mathrm{div}(\boldsymbol{v}) = 0\\
&\frac{\mathrm{d}p}{\mathrm{d}t} + A(p,\tau)\mathrm{div}(\boldsymbol{v}) = B(p,\tau)(\lambda(\mathrm{div}(\boldsymbol{v}))^2 + 2\mu D{:}D + (\nabla\kappa)(\nabla T) + \kappa\triangle T)
\end{aligned} \tag{14}$$

with functions

$$A = \frac{\tau(\varepsilon_\tau + p)}{\varepsilon_p} \qquad B = \frac{\tau}{\varepsilon_p}.$$

Note that the internal energy and entropy can be expressed through the functions $A = A(p,\tau)$, $B = B(p,\tau)$ by the formulae

$$\varepsilon_p = \frac{\tau}{B} \qquad \varepsilon_\tau = \frac{A}{B} - p \qquad \eta_p = \frac{\tau}{BT} \qquad \eta_\tau = \frac{A}{BT}.$$

The conditions $\varepsilon_{p\tau} = \varepsilon_{\tau p}$, $\eta_{p\tau} = \eta_{\tau p}$ lead to the restrictions

$$\tau B_\tau + BA_p - AB_p = B^2 + B \qquad \tau T_\tau = AT_p - TB. \tag{15}$$

In the case of an ideal gas (i.e. the gas that obeys the Clapeyron equation $T = R^{-1}p\tau$) $B = B(\tau p)$, $A = p(1 + B(\tau p))$ with an arbitrary function $B(\tau p)$. For a polytropic gas $\varepsilon = (\gamma-1)^{-1}\tau p$ and once more this simplifies the functions A and B: $B = (\gamma - 1)$, $A = \gamma p$. Here R is the gas constant and γ is a polytropic exponent. If τ and μ are constants, then system (14) is split into two parts: the Navier–Stokes equations and energy equation[3].

[3] Sometimes in the literature equations (14) are called the full Navier–Stokes equations.

3.1. Two-dimensional steady viscous gas dynamics equations

In this paper we study the two-dimensional steady viscous gas dynamics equations

$$
\begin{aligned}
&u\tau_x + v\tau_y - \tau\left(u_x + v_y + \nu\frac{u}{x}\right) = 0\\
&uu_x + vu_y + \tau p_x = \tau\Bigg((\lambda+\mu)\left(u_x + v_y + \nu\frac{u}{x}\right)_x + \lambda_x\left(u_x + v_y + \nu\frac{u}{x}\right)\\
&\qquad + 2\mu_x u_x + \mu_y(u_y + v_x) + \mu\left(u_{xx} + u_{yy} + \nu\frac{u_x}{x}\right) - \mu\nu\frac{u}{x^2}\Bigg)\\
&uv_x + vv_y + \tau p_y = \tau\Bigg((\lambda+\mu)\left(u_x + v_y + \nu\frac{u}{x}\right)_y + \lambda_y\left(u_x + v_y + \nu\frac{u}{x}\right)\\
&\qquad + \mu_x(u_y + v_x) + 2\mu_y v_y + \mu\left(v_{xx} + v_{yy} + \nu\frac{v_x}{x}\right)\Bigg)\\
&up_x + vp_y + A(p,\rho)\left(u_x + v_y + \nu\frac{u}{x}\right) = B(p,\rho)\Bigg(\kappa\left(T_{xx} + T_{yy} + \nu\frac{T_x}{x}\right) + \kappa_x T_x + \kappa_y T_y\\
&\qquad + \mu\left(2\left(u_x^2 + v_y^2 + \nu\frac{u^2}{x^2}\right) + (u_y + v_x)^2\right)\Bigg)
\end{aligned}
\tag{16}
$$

where $\nu = 0$ corresponds to the plane flows and $\nu = 1$ to the axisymmetrical flows. The case of ideal gas $T = R^{-1}p\tau$ where the first, $\lambda = \lambda(T)$, and second, $\mu = \mu(T)$, coefficients of viscosity are related by the equation $\lambda = -2\mu/3$ and $\kappa = \kappa(T)$ has been studied in [12]. Here we study the gas dynamics equations with arbitrary state equations.

Since the arbitrary elements satisfy restrictions (15) and $A = A(p,\tau)$, $B = B(p,\tau)$, $\lambda = \lambda(p,\tau)$, $\mu = \mu(p,\tau)$, $\kappa = \kappa(p,\tau)$, $T = T(p,\tau)$, hence for calculating the equivalence group of transformations we have to append the equations

$$
\begin{array}{llll}
A_x = 0 & A_y = 0 & A_u = 0 & A_v = 0\\
B_x = 0 & B_y = 0 & B_u = 0 & B_v = 0\\
\lambda_x = 0 & \lambda_y = 0 & \lambda_u = 0 & \lambda_v = 0\\
\mu_x = 0 & \mu_y = 0 & \mu_u = 0 & \mu_v = 0\\
\kappa_x = 0 & \kappa_y = 0 & \kappa_u = 0 & \kappa_v = 0\\
T_x = 0 & T_y = 0 & T_u = 0 & T_v = 0
\end{array}
\tag{17}
$$

to equations (16). All coefficients of the infinitesimal generator of the equivalence group are dependent on all independent, dependent variables and arbitrary elements

$$x, y, u, v, \tau, p, A, B, \lambda, \mu, \kappa, T.$$

All necessary calculations were carried out on a computer using the symbolic manipulation program REDUCE [20]. The calculations showed that the group of equivalence transformations of equations (16), (17) corresponds to the Lie algebra with generators

$$X_1^e = \partial_y \qquad X_2^e = \partial_p \qquad X_3^e = x\partial_x + y\partial_y + \lambda\partial_\lambda + \mu\partial_\mu + \kappa\partial_\kappa$$

$$X_4^e = x\partial_x + y\partial_y + u\partial_u + v\partial_v + 2\tau\partial_\tau + 2\kappa\partial_\kappa \qquad X_5^e = -\tau\partial_\tau + p\partial_p + A\partial_A + \lambda\partial_\lambda + \mu\partial_\mu + \kappa\partial_\kappa.$$

In the case $\nu = 0$ there are two more generators

$$X_6^e = \partial_x \qquad X_7^e = y\partial_x - x\partial_y + v\partial_u - u\partial_v$$

which correspond to shift and rotation.

Remark. If instead of the functions $A(p, \tau)$ and $B(p, \tau)$, one considers the internal energy $\varepsilon(p, \tau)$, then the operators X_2^e, X_4^e and X_5^e are changed to

$$X_2^e = \partial_p - \tau\partial_\varepsilon \qquad X_4^e = x\partial_x + y\partial_y + u\partial_u + v\partial_v + 2\tau\partial_\tau + 2\kappa\partial_\kappa + 2\varepsilon\partial_\varepsilon$$

$$X_5^e = -\tau\partial_\tau + p\partial_p + \lambda\partial_\lambda + \mu\partial_\mu + \kappa\partial_\kappa$$

and there is one more generator $X_8^e = \partial_\varepsilon$.

3.2. Admitted group

For finding the admitted group we look for the generator

$$X = \zeta^x\partial_x + \zeta^y\partial_y + \zeta^u\partial_u + \zeta^v\partial_v + \zeta^\tau\partial_\tau + \zeta^p\partial_p$$

with the coefficients depending on x, y, u, v, τ, p. Calculations lead to the following result.

The kernel of the fundamental Lie algebra is made up of the generator

$$X_1 = \partial_y$$

if $\nu = 1$ and

$$X_1 = \partial_y \qquad X_2 = \partial_x \qquad X_3 = y\partial_x - x\partial_y + v\partial_u - u\partial_v$$

if $\nu = 0$. An extension of the kernel of the principal Lie algebra occurs by specializing the functions $A = A(p, \tau), B = B(p, \tau), \lambda = \lambda(p, \tau), \mu = \mu(p, \tau), \kappa = \kappa(p, \tau), T = T(p, \tau)$. Note that the functions $A = A(p, \tau)$, $B = B(p, \tau)$, $T = T(p, \tau)$ have to satisfy equations (15). There are three types of generators admitted by system (16). Further, α, β and δ are arbitrary constants.

Type (a). If the functions $A(\tau, p)$, $B(\tau, p)$, $\lambda(\tau, p)$, $\mu(\tau, p)$, $\kappa(\tau, p)$, $T(\tau, p)$ satisfy the equations

$$\begin{array}{ll} \alpha\tau A_\tau + A_p = 0 & \alpha\tau B_\tau + B_p = 0 \\ \alpha\tau\mu_\tau + \mu_p = \beta\mu & \alpha\tau\lambda_\tau + \lambda_p = \beta\lambda \\ \alpha\tau T_\tau + T_p = \delta T & \alpha\tau\kappa_\tau + \kappa_p = (-\delta + \alpha + \beta)\kappa \end{array} \tag{18}$$

then there is one more admitted generator:

$$Y_a = \alpha(u\partial_u + v\partial_v) + 2\alpha\tau\partial_\tau + 2\partial_p + (\alpha + 2\beta)(x\partial_x + y\partial_y).$$

The general solution of equations (18) is

$$\begin{array}{lll} A = A(\tau e^{-\alpha p}) & B = B(\tau e^{-\alpha p}) & \mu = e^{\beta p} M(\tau e^{-\alpha p}) \\ \lambda = e^{\beta p}\Lambda(\tau e^{-\alpha p}) & T = e^{\delta p}\Theta(\tau e^{-\alpha p}) & \kappa = e^{(-\delta+\alpha+\beta)p} K(\tau e^{-\alpha p}) \end{array} \tag{19}$$

where the functions $A(z)$, $B(z)$ and $\Theta(z)$ satisfy the equations ($z \equiv \tau e^{-\alpha p}$):

$$-\alpha z B A' + z B'(1 + \alpha A) = B^2 + B \qquad (1 + \alpha A) z\Theta' = (\delta A - B)\Theta. \tag{20}$$

The internal energy is represented by the formula

$$\varepsilon = e^{\alpha p}(\varphi(z) - zp) + \psi(p) \qquad \psi'(p) = C e^{\alpha p}$$

where the function $\varphi(z)$ and constant C can be accounted arbitrarily and they are related to the functions $A(z)$ and $B(z)$ by the formulae

$$\varphi'(z) = \frac{A(z)}{B(z)} \qquad C = z + \frac{z}{B(z)} + \alpha z\varphi'(z) - \alpha\varphi(z).$$

In this case the function $\Theta(z)$ has to satisfy the equation

$$(C - z + \alpha\varphi(z))\,\Theta'(z) = (\delta\varphi'(z) - 1)\Theta(z).$$

Type (b). If the functions $A(\tau, p)$, $B(\tau, p)$, $\lambda(\tau, p)$, $\mu(\tau, p)$, $\kappa(\tau, p)$, $T(\tau, p)$ satisfy the equations

$$\begin{array}{ll} \alpha\tau A_\tau + pA_p = A & \alpha\tau B_\tau + pB_p = 0 \\ \alpha\tau\mu_\tau + p\mu_p = (\beta+1)\mu & \alpha\tau\lambda_\tau + p\lambda_p = (\beta+1)\lambda \\ \alpha\tau T_\tau + pT_p = \delta T & \alpha\tau\kappa_\tau + pT_p = (-\delta+2+\alpha+\beta)\kappa \end{array} \tag{21}$$

then there is an extension by the generator

$$Y_b = (1+\alpha)(u\partial_u + v\partial_v) + 2\alpha\tau\partial_\tau + 2p\partial_p + (\alpha+2\beta+1)(x\partial_x + y\partial_y).$$

The general solution of equations (21) is

$$\begin{array}{lll} A = p\hat{A}(\tau p^{-\alpha}) & B = B(\tau p^{-\alpha}) & \mu = p^{\beta+1}M(\tau p^{-\alpha}) \\ \lambda = p^{\beta+1}\Lambda(\tau p^{-\alpha}) & T = p^{\delta}\Theta(\tau p^{-\alpha}) & \kappa = p^{-\delta+\alpha+\beta+2}K(\tau p^{-\alpha}) \end{array} \tag{22}$$

where the functions $\hat{A}(z)$, $B(z)$ and $\Theta(z)$ satisfy the equations ($z \equiv \tau p^{-\alpha}$):

$$-\alpha zB\hat{A}' + zB'(1+\alpha\hat{A}) = B^2 + B - B\hat{A} \qquad (1+\alpha\hat{A})z\Theta' = (\delta\hat{A} - B)\Theta. \tag{23}$$

The internal energy is represented by the formula

$$\varepsilon = p^{(\alpha+1)}(\varphi(z) - z) + \psi(p) \qquad \psi'(p) = Cp^{\alpha}$$

where the function $\varphi(z)$ and constant C are arbitrary and they are related to the functions $\hat{A}(z)$ and $B(z)$ by the formulae

$$\varphi'(z) = \frac{\hat{A}(z)}{B(z)} \qquad C = z + \frac{z}{B(z)} + \alpha z\varphi'(z) - (\alpha+1)\varphi(z).$$

The function $\Theta(z)$ is represented through the function $\varphi(z)$ by the formula

$$(C - z + (\alpha+1)\varphi(z))\,\Theta'(z) = (\delta\varphi'(z) - 1)\Theta(z).$$

Note that an ideal gas belongs to this type if $\delta = \alpha + 1$ and the function $\varphi(z)$ satisfies the equation

$$\delta(z\varphi' - \varphi) = C.$$

Type (c). If the functions $A(\tau, p)$, $B(\tau, p)$, $\lambda(\tau, p)$, $\mu(\tau, p)$, $\kappa(\tau, p)$, $T(\tau, p)$ satisfy the equations

$$\begin{array}{llll} A_\tau = 0 & B_\tau = 0 & \tau\mu_\tau = \beta\mu & \tau\lambda_\tau = \beta\lambda \\ \tau T_\tau = \delta T & \tau\kappa_\tau = (-\delta+1+\beta)\kappa & & \end{array} \tag{24}$$

then there is one more admitted generator:

$$Y_c = u\partial_u + v\partial_v + 2\tau\partial_\tau + (1+2\beta)(x\partial_x + y\partial_y).$$

The general solution of equations (24) is

$$\begin{array}{llll} A = A(p) & B = B(p) & \mu = \tau^{\beta}M(p) & \lambda = \tau^{\beta}\Lambda(p) \\ T = \tau^{\delta}\Theta(p) & \kappa = \tau^{-\delta+\beta+1}K(p) & & \end{array} \tag{25}$$

where the functions $A(p)$, $B(p)$ and $\Theta(p)$ satisfy the equations

$$BA' - AB' = B^2 + B \qquad A\Theta' = (\delta + B)\Theta. \tag{26}$$

The internal energy is represented by the formula

$$\varepsilon = \tau\varphi(p) - \tau p$$

	λ	μ	T	κ	A	B	z	Condition
a	$e^{\beta p}\Lambda(z)$	$e^{\beta p}M(z)$	$e^{\delta p}\Theta(z)$	$e^{(-\delta+\alpha+\beta)p}K(z)$	$A(z)$	$B(z)$	$\tau e^{-\alpha p}$	(20)
b	$p^{\beta+1}\Lambda(z)$	$p^{\beta+1}M(z)$	$p^{\delta}\Theta(z)$	$p^{-\delta+\alpha+\beta+2}K(z)$	$p\hat{A}(z)$	$B(z)$	$\tau p^{-\alpha}$	(23)
c	$\tau^{\beta}\Lambda(p)$	$\tau^{\beta}M(p)$	$\tau^{\delta}\Theta(p)$	$\tau^{-\delta+\beta+1}K(p)$	$A(p)$	$B(p)$	p	(26)

where $\varphi(p)$ is an arbitrary function and is related to the functions $A(p)$ and $B(p)$ by the formula

$$\varphi(p) = \frac{A(p)}{B(p)}.$$

In this case the function $\Theta(p)$ is related to the function $\varphi(z)$ by the formula

$$\varphi(p)\Theta'(p) = (1 - \delta + \delta\varphi'(p))\Theta(p).$$

Note that if $\delta = 1$ and $\varphi = Cp$, then the gas is ideal.

The final results of the group classification are presented in table 1.

In this table the first column means the type of extension of the algebra $\{X\}$ or $\{X, Y\}$: the type a, b, or c, respectively. The last column means conditions for the state functions.

Thus, there are three kinds of extensions of the groups admitted by equations (16), which depend on the specifications of the functions $A = A(p,\tau)$, $B = B(p,\tau)$, $\lambda = \lambda(p,\tau)$, $\mu = \mu(p,\tau)$, $\kappa = \kappa(p,\tau)$, $T = T(p,\tau)$. These extensions can be one dimensional and two dimensional[4].

The one-dimensional extensions are with the generators $\{Y_a\}$, $\{Y_b\}$ or $\{Y_c\}$.

The two-dimensional extensions are with the generators $\{Y_a, Y_b\}$, $\{Y_a, Y_c\}$ or $\{Y_b, Y_c\}$.

The group with the extension $\{Y_a, Y_b\}$ is admitted by equations (16) if

$$A = A_0\tau^{\alpha} \qquad B = -1 \qquad \mu = \mu_0\tau^{\beta+\alpha} \qquad \lambda = \lambda_0\tau^{\beta+\alpha}$$
$$\kappa = \kappa_0\tau^{\beta+2\alpha} \qquad T = T_0\tau \qquad \alpha \neq 0.$$

In this case the internal energy is $\varepsilon = -\left(\tau p + A_0\int\tau^{\alpha}\,d\tau\right)$. Instead of the operators Y_a and Y_b, one can use their linear combinations:

$$\hat{Y}_a = \partial_p \qquad \hat{Y}_b = (1+\alpha)(u\partial_u + v\partial_v) + 2\tau\partial_\tau + (\alpha + 2\beta + 1)(x\partial_x + y\partial_y).$$

The group with the extension of type $\{Y_a, Y_c\}$ is admitted by equations (16) if

$$A = A_0 \qquad B = -1 \qquad \mu = \mu_0\tau^{\beta}e^{\alpha p} \qquad \lambda = \lambda_0\tau^{\beta}e^{\alpha p}$$
$$\kappa = \kappa_0\tau^{\beta - A_0\sigma}e^{(\alpha-\sigma)p} \qquad T = T_0\tau^{1+A_0\sigma}e^{\sigma p}.$$

In this case the internal energy is $\varepsilon = -(\tau p + A_0\tau)$ and by taking linear combinations of the operators Y_a and Y_c, one obtains another basis of the generators:

$$\hat{Y}_a = \partial_p + \alpha(x\partial_x + y\partial_y) \qquad \hat{Y}_c = u\partial_u + v\partial_v + 2\tau\partial_\tau + (2\beta + 1)(x\partial_x + y\partial_y).$$

There is a third type of extensions $\{Y_b, Y_c\}$ if

$$A = \gamma p \qquad B = \gamma - 1 \qquad \mu = \mu_0\tau^{\beta}p^{1+\alpha} \qquad \lambda = \lambda_0\tau^{\beta}p^{1+\alpha}$$
$$\kappa = \kappa_0\tau^{\gamma(1-\alpha)+\beta}p^{\alpha-\delta+2} \qquad T = T_0\tau^{\gamma(\delta-1)+1}p^{\delta} \qquad \gamma \neq 1.$$

The internal energy in this case is

$$\varepsilon = \frac{\tau p}{\gamma - 1}$$

[4] There is no three-dimensional extension because of incompatibility of the system of differential equations for the functions $A = A(p,\tau)$, $B = B(p,\tau)$, $\lambda = \lambda(p,\tau)$, $\mu = \mu(p,\tau)$, $\kappa = \kappa(p,\tau)$, $T = T(p,\tau)$.

and linear combinations of the operators Y_b and Y_c are

$$\hat{Y}_b = u\partial_u + v\partial_v + 2p\partial_p + (2\alpha+1)(x\partial_x + y\partial_y) \qquad \hat{Y}_c = \tau\partial_\tau - p\partial_p + (\beta-\alpha)(x\partial_x + y\partial_y).$$

Note that a polytropic gas belongs to the last case of gases, where γ is a polytropic exponent.

In the formulae above $A_0, \mu_0, \lambda_0, \kappa_0, T_0, \alpha, \beta, \gamma, \delta, \sigma$ are arbitrary constants; the commutators are

$$[\hat{Y}_a, \hat{Y}_b] = 0 \qquad [\hat{Y}_a, \hat{Y}_c] = 0 \qquad [\hat{Y}_b, \hat{Y}_c] = 0.$$

Remark. By direct checking one can set for the general unsteady three-dimensional gas flow the same models of types (a), (b) and (c), described by equations (19), (20), (22), (23), (25) and (26), with the following generalized generators:

$$Y_a = \alpha \boldsymbol{v}\partial_{\boldsymbol{v}} + 2\alpha\tau\partial_\tau + 2\partial_p + (\alpha+2\beta)\boldsymbol{x}\partial_{\boldsymbol{x}} + 2\beta t\partial_t$$

$$Y_b = (1+\alpha)\boldsymbol{v}\partial_{\boldsymbol{v}} + 2\alpha\tau\partial_\tau + 2p\partial_p + (\alpha+2\beta+1)\boldsymbol{x}\partial_{\boldsymbol{x}} + 2\beta t\partial_t$$

$$Y_c = \boldsymbol{v}\partial_{\boldsymbol{v}} + 2\tau\partial_\tau + (1+2\beta)\boldsymbol{x}\partial_{\boldsymbol{x}} + 2\beta t\partial_t.$$

The kernel includes the Galilean group with generators

$$\boldsymbol{X}_i = \partial_{x_i} \qquad \boldsymbol{X}_{3+i} = t\partial_{x_i} + \partial_{v_i} \qquad \boldsymbol{Y}_{ij} = x_i\partial_{x_j} - x_j\partial_{x_i} + v_i\partial_{v_j} - v_j\partial_{v_i}$$

$$\boldsymbol{X}_{10} = \partial_t \qquad (i, j = 1, 2, 3 \quad i < j).$$

It has to be mentioned that the group classification of the viscous gas dynamics equations in the case of an ideal gas with the first, $\lambda = \lambda(T)$, and second, $\mu = \mu(T)$, coefficients of viscosity related by the equation $\lambda = -2\mu/3$, and $\kappa = \kappa(T)$ was done in [11]. Two-dimensional steady viscous gas dynamics equations and their simplifications (parabolized models) for ideal gas were studied in [12]. The group classification of spherically symmetric flows with arbitrary state equations was considered in [14].

4. Optimal system of subalgebras

In this section two groups are studied. One is the group with generators

$$L_4 = \{X_1, X_2, X_3, Y\}.$$

The other is the group with generators

$$L_2 = \{X_2, Y\}.$$

Here Y is one of the generators: $Y = Y_a$ (with the parameter z, which is used later $z = \alpha+2\beta$), $Y = Y_b$ ($z = \alpha+2\beta+1$) or $Y = Y_c$ ($z = 2\beta+1$). These groups correspond to the plane ($\nu = 0$) case and axisymmetrical ($\nu = 1$) case with one extension, respectively. The classifications of subalgebras of the algebras L_4 and L_2 are given in this section.

The classification subdivides a set of H-solutions into equivalent (similar) classes. Any two H-solutions f_1 and f_2 are elements of the same equivalence class if there exists a transformation $T_a \in GS$ such that $f_2 = T_a f_1$. Otherwise f_1, f_2 belong to different classes and they are called essentially different H-solutions. The classification of H-solutions is related to the optimal system of subalgebras Θ of the admitted algebra L. To obtain the optimal system of subalgebras Θ, we use the algorithm developed in [21, 22]. Let us consider the algebra

$L_4 = \{X_1, X_2, X_3, Y\}$. The table of commutators is

	X_1	X_2	X_3	Y
X_1	0	0	X_2	zX_1
X_2	0	0	$-X_1$	zX_2
X_3	$-X_2$	X_1	0	0
Y	$-zX_1$	$-zX_2$	0	0

Automorphisms are recovered by the table of commutators and consist of the automorphisms

$$A_1:\quad x_1' = x_1 + zy_1a_1 \qquad x_2' = x_2 - x_3a_1$$
$$A_2:\quad x_1' = x_1 + x_3a_2 \qquad x_2' = x_2 + zy_1a_2$$
$$A_3:\quad x_1' = x_1\cos a_3 - x_2\sin a_3 \qquad x_2' = x_1\sin a_3 + x_2\cos a_3$$
$$A_4:\quad x_1' = x_1\mathrm{e}^{za_4} \qquad x_2' = x_2\mathrm{e}^{za_4}.$$

Here x_i $(i = 1, 2, 3)$ and y_1 are coordinates of the operator $Z = x_1X_1 + x_2X_2 + x_3X_3 + y_1Y$ before the transformation and x_i' $(i = 1, 2, 3)$ and y_1' are coordinates of the operator Z' after action of the automorphism, a_i are parameters of the automorphisms. In the expressions for automorphisms only transformed coordinates are presented. There is also one involution

$$E:\quad x_1' = -x_1 \qquad x_2' = -x_2$$

which corresponds to the change of variables $x \to -x$, $y \to -y$, $u \to -u$ and $v \to -v$ without change of equations (16).

The Lie algebra L_4 has the following decomposition: $L_4 = N_2 \oplus J_2$, where $N_2 = \{X_3, Y\}$ is a subalgebra and $J_2 = \{X_1, X_2\}$ is an ideal. The Lie algebra N_2 is Abelian. Hence, its classification is trivial and consists of the subalgebras

$$\{X_3 + hY\}, \qquad \{Y\}, \qquad \{X_3, Y\}.$$

The optimal system of subalgebras of the algebra L_4 is obtained by gluing the ideal J_2 to the constructed subalgebras of the optimal system of subalgebras of the algebra N_2.

Because the number of independent variables is two, invariant solutions can be constructed only with respect to one- and two-dimensional subalgebras. These subalgebras of the optimal system are

$$\{X_3, Y\}, \qquad \{X_1, Y + hX_2\}, \qquad \{X_1, X_2\}, \qquad \{X_3 + qY\}, \qquad \{Y + hX_2\}, \qquad \{X_1\}$$

where $zh = 0$, and q and h are arbitrary constants.

The optimal system of subalgebras of the algebra $L_2 = \{X_2, Y\}$ consists of the subalgebras

$$\{X_2, Y\}, \qquad \{Y + hX_2\}, \qquad \{X_2\}$$

where $zh = 0$ and h is an arbitrary constant.

4.1. Representations of invariant solutions

The next step in the construction of invariant solutions consists of finding universal invariants. Note that the subalgebras from the optimal system of the algebra L_2 are those from the optimal system of the algebra L_4. Thus, it is enough to consider representations of invariant solutions of the algebra L_4. Before presenting the results, we give some remarks. In the case of two-dimensional subalgebras, the representations of invariant solutions are obtained by assuming that all invariants are constants. The subalgebra $\{X_1, X_2\}$ has no invariant solutions.

The invariant solution with respect to the subalgebra $\{X_1\}$ is a trivial one-dimensional steady solution of the viscous gas dynamics equations.

According to the theory of group analysis [18], after constructing the representations of invariant solutions one needs to substitute the representations of solutions into the original system of equations.

4.1.1. Subalgebra $\{X_3, Y\}$. For the operator X_3 it is convenient to use the cylindrical coordinates

$$x = r\cos\theta \qquad y = r\sin\theta \qquad u = U\cos\theta - V\sin\theta \qquad v = U\cos\theta + V\cos\theta.$$

In these coordinates there are the following relations:

$$X_3 = \partial_\theta \qquad x\partial_x + y\partial_y = r\partial_r \qquad u\partial_u + v\partial_v = U\partial_U + V\partial_V.$$

Note that if $z = 0$, then there are no invariant solutions. Hence, we have only to study the case $z \neq 0$.

In case (a) $z = \alpha + 2\beta \neq 0$ and the universal invariant consists of the invariants

$$Ur^{-\alpha/z},\ Vr^{-\alpha/z}, \tau r^{-2\alpha/z}, p - 2z^{-1}\ln r.$$

In case (b) $z = \alpha + 2\beta + 1 \neq 0$ and the universal invariant consists of the invariants

$$Ur^{-(1+\alpha)/z},\ Vr^{-(1+\alpha)/z}, \tau r^{-2\alpha/z}, pr^{-2/z}.$$

In case (c) $z = 2\beta + 1 \neq 0$ and the universal invariant consists of the invariants

$$Ur^{-1/z}, Vr^{-1/z}, \tau r^{-2/z}, p.$$

4.1.2. Subalgebra $\{X_1, Y + hX_2\}$ $(zh = 0)$. If $z = 0$ and $h = 0$, then there are no invariant solutions.

In case (a) $z = \alpha + 2\beta$. If $z \neq 0$, then $h = 0$ and the universal invariant consists of the invariants

$$uy^{-\alpha/z}, vy^{-\alpha/z}, \tau y^{-2\alpha/z}, p - 2z^{-1}\ln y.$$

If $z = \alpha + 2\beta = 0$, then there is an invariant solution only if $h \neq 0$ and the universal invariant is

$$u\mathrm{e}^{-\alpha y/h}, v\mathrm{e}^{-\alpha y/h}, \tau\mathrm{e}^{-2\alpha y/h}, p - 2h^{-1}y.$$

In case (b) $z = \alpha + 2\beta + 1$. If $z \neq 0$, then $h = 0$ and the universal invariant consists of the invariants

$$uy^{-(1+\alpha)/z}, vy^{-(1+\alpha)/z}, \tau y^{-2\alpha/z}, py^{-2/z}.$$

If $z = \alpha + 2\beta = 0$, then there is an invariant solution only if $h \neq 0$ and the universal invariant is

$$u\mathrm{e}^{-(1+\alpha)y/h}, v\mathrm{e}^{-(1+\alpha)y/h}, \tau\mathrm{e}^{-2\alpha y/h}, p\mathrm{e}^{-2y/h}.$$

In case (c) $z = 2\beta + 1$. If $z \neq 0$, then $h = 0$ and the universal invariant consists of the invariants

$$uy^{-1/z}, vy^{-1/z}, \tau y^{-2/z}, p.$$

If $z = \alpha + 2\beta = 0$, then there is an invariant solution only if $h \neq 0$ and the universal invariant is

$$u\mathrm{e}^{-y/h}, v\mathrm{e}^{-y/h}, \tau\mathrm{e}^{-2y/h}, p.$$

4.1.3. Subalgebra $\{X_3 + qY\}$. The operators $X_3 + qY$ in cylindrical coordinates are

$$X_3 + qY_a = \partial_\theta + q((\alpha + 2\beta)r\partial_r + \alpha(U\partial_U + V\partial_V) + 2\alpha\tau\partial_\tau + 2\partial_p)$$

$$X_3 + qY_b = \partial_\theta + q((\alpha + 2\beta + 1)r\partial_r + (1 + \alpha)(U\partial_U + V\partial_V) + 2\alpha\tau\partial_\tau + 2p\partial_p)$$

$$X_3 + qY_c = \partial_\theta + q((2\beta + 1)r\partial_r + U\partial_U + V\partial_V + 2\tau\partial_\tau).$$

In case (a) the universal invariant consists of the invariants

$$\bar{J} = (U\mathrm{e}^{-\alpha q\theta}, V\mathrm{e}^{-\alpha q\theta}, \tau\mathrm{e}^{-2\alpha q\theta}, p - 2q\theta) \qquad \bar{\bar{J}} = r\mathrm{e}^{-(\alpha+2\beta)q\theta}.$$

In case (b) the universal invariant consists of the invariants

$$\bar{J} = \left(U\mathrm{e}^{-(1+\alpha)q\theta}, V\mathrm{e}^{-(1+\alpha)q\theta}, \tau\mathrm{e}^{-2\alpha q\theta}, p\mathrm{e}^{-2q\theta}\right) \qquad \bar{\bar{J}} = r\mathrm{e}^{-(\alpha+2\beta+1)q\theta}.$$

In case (c) the universal invariant consists of the invariants

$$\bar{J} = (U\mathrm{e}^{-q\theta}, V\mathrm{e}^{-q\theta}, \tau\mathrm{e}^{-2q\theta}, p) \qquad \bar{\bar{J}} = r\mathrm{e}^{-(2\beta+1)q\theta}.$$

4.1.4. Subalgebra $\{Y + hX_2\}$ $(zh = 0)$. Note that if $z = 0$ and $h = 0$, then there are no invariant solutions.

In case (a) $z = \alpha + 2\beta$. If $z \neq 0$, then $h = 0$ and the universal invariant consists of the invariants

$$\bar{J} = \left(uy^{-\alpha/z}, vy^{-\alpha/z}, \tau y^{-2\alpha/z}, p - 2z^{-1}\ln y\right) \qquad \bar{\bar{J}} = x/y.$$

If $z = \alpha + 2\beta = 0$, then there is an invariant solution only if $h \neq 0$ and the universal invariant is

$$\bar{J} = \left(u\mathrm{e}^{-\alpha y/h}, v\mathrm{e}^{-\alpha y/h}, \tau\mathrm{e}^{-2\alpha y/h}, p - 2h^{-1}y\right) \qquad \bar{\bar{J}} = x.$$

In case (b) $z = \alpha + 2\beta + 1$. If $z \neq 0$, then $h = 0$ and the universal invariant consists of the invariants

$$\bar{J} = \left(uy^{-(1+\alpha)/z}, vy^{-(1+\alpha)/z}, \tau y^{-2\alpha/z}, py^{-2/z}\right) \qquad \bar{\bar{J}} = x/y.$$

If $z = 2\beta + 1 = 0$, then there is an invariant solution only if $h \neq 0$ and the universal invariant is

$$\bar{J} = \left(u\mathrm{e}^{-(1+\alpha)y/h}, v\mathrm{e}^{-(1+\alpha)y/h}, \tau\mathrm{e}^{-2\alpha y/h}, p\mathrm{e}^{-2y/h}\right) \qquad \bar{\bar{J}} = x.$$

In case (c) $z = 2\beta + 1$. If $z \neq 0$, then $h = 0$ and the universal invariant consists of the invariants

$$\bar{J} = \left(uy^{-1/z}, vy^{-1/z}, \tau y^{-2/z}, p\right) \qquad \bar{\bar{J}} = x/y.$$

If $z = \alpha + 2\beta = 0$, then there is an invariant solution only if $h \neq 0$ and the universal invariant is

$$\bar{J} = \left(u\mathrm{e}^{-y/h}, v\mathrm{e}^{-y/h}, \tau\mathrm{e}^{-2y/h}, p\right) \qquad \bar{\bar{J}} = x.$$

5. Invariant solutions

In this section we demonstrate the construction of reduced systems for invariant solutions. As an example the subalgebra $\{Y + hX_2\}$ for an ideal gas $(T = R^{-1}p\tau)$ of type (c) is taken. Note that in the case of an ideal gas $B = \gamma - 1$, $A = \gamma p$, where γ is a constant. The obtained reduced systems are systems of ordinary differential equations. For solving the ordinary differential equations, one can use well-developed numerical methods.

5.1. The case $2\beta + 1 \neq 0$

At first, let us consider $z = 2\beta + 1 \neq 0$. In this case the representation of the invariant solution is

$$u = U(\xi)y^q \qquad v = V(\xi)y^q \qquad \tau = G(\xi)y^{2q} \qquad p = P(\xi) \qquad \xi = \frac{x}{y}$$

where $q = z^{-1}$. The next step in obtaining the reduced system is the substitution of the representation of the invariant solution into the initial system of viscous gas dynamics equations. For example, the equation of mass conservation becomes

$$\frac{\mathrm{d}G}{\mathrm{d}\xi}(U - \xi V) - G\frac{\mathrm{d}U}{\mathrm{d}\xi}(U - \xi V) + (q-1)GV = 0.$$

If $q = 1$ (or $\beta = 0$), then the last equation can be integrated:

$$U = \xi V + c_1 G$$

where c_1 is an arbitrary constant. For the sake of simplicity, we present the reduced system for this case ($\beta = 0$) and also assume that the functions $\Lambda(p)$, $M(p)$, $K(p)$ are constants. The remaining equations in this case are

$$\begin{aligned}
&\frac{\mathrm{d}G}{\mathrm{d}\xi}c_1^2 + c_1\left(2V - (\xi^2+1)\,\mathrm{Re}^{-1}(\tilde{\lambda}+2)\frac{\mathrm{d}^2G}{\mathrm{d}\xi^2}\right) + (\xi^2+1)\left(\frac{\mathrm{d}P}{\mathrm{d}\xi} - 2(\tilde{\lambda}+2)\,\mathrm{Re}^{-1}\frac{\mathrm{d}V}{\mathrm{d}\xi}\right) = 0\\
&c_1\left(\mathrm{Re}^{-1}\frac{\mathrm{d}^2G}{\mathrm{d}\xi^2}\xi(\tilde{\lambda}+1) + \frac{\mathrm{d}V}{\mathrm{d}\xi}\right) + 2\,\mathrm{Re}^{-1}\frac{\mathrm{d}V}{\mathrm{d}\xi}\xi(\tilde{\lambda}+1) - \frac{\mathrm{d}P}{\mathrm{d}\xi}\xi\\
&\qquad - \mathrm{Re}^{-1}\frac{\mathrm{d}^2V}{\mathrm{d}\xi^2}(\xi^2+1) + G^{-1}V^2 = 0\\
&2(\gamma-1)^{-1}\,\mathrm{Re}\,PV + c_1^2\left(2\frac{\mathrm{d}G}{\mathrm{d}\xi}G\xi - \left(\frac{\mathrm{d}G}{\mathrm{d}\xi}\right)^2(\tilde{\lambda}+2+\xi^2) - G^2\right)\\
&\qquad + c_1\left(\frac{\mathrm{d}G}{\mathrm{d}\xi}(\mathrm{Re}\,P - 4V(\tilde{\lambda}+1)) + 2\frac{\mathrm{d}V}{\mathrm{d}\xi}G(\xi^2-1) - 2\frac{\mathrm{d}G}{\mathrm{d}\xi}\frac{\mathrm{d}V}{\mathrm{d}\xi}\xi(\xi^2+1)\right)\\
&\qquad + c_1(\gamma-1)^{-1}\,\mathrm{Re}\left(\frac{\mathrm{d}G}{\mathrm{d}\xi}P + \frac{\mathrm{d}P}{\mathrm{d}\xi}G\right) + 2V(\mathrm{Re}\,P - 2V(\tilde{\lambda}+1))\\
&\qquad - \left(\frac{\mathrm{d}V}{\mathrm{d}\xi}\right)^2(\xi^2+1)^2 + \frac{\gamma\,\mathrm{Pr}^{-1}}{(\gamma-1)}\left(2\left(\frac{\mathrm{d}G}{\mathrm{d}\xi}P\xi + \frac{\mathrm{d}P}{\mathrm{d}\xi}G\xi - GP\right)\right.\\
&\qquad \left.-\ (\xi^2+1)\left(2\frac{\mathrm{d}G}{\mathrm{d}\xi}\frac{\mathrm{d}P}{\mathrm{d}\xi} + \frac{\mathrm{d}^2G}{\mathrm{d}\xi^2}P + \frac{\mathrm{d}^2P}{\mathrm{d}\xi^2}G\right)\right) = 0.
\end{aligned} \tag{27}$$

Here Re is the Reynolds number and Pr is the Prandtl number. The nondimensional dependent variables $\tilde{G}$, $\tilde{V}$, $\tilde{P}$, $\tilde{c}_1$ are related to the dimensional variables G, V, P, c_1 by the formulae

$$G = L^{-2}\tau_0\tilde{G} \qquad V = Lv_0\tilde{V} \qquad P = p_0\tilde{P} \qquad \lambda = \mu\tilde{\lambda} \qquad c_1 = v_0L\tau_0^{-1}\tilde{c}_1$$

$$p_0 = v_0^2\tau_0^{-1} \qquad \mathrm{Pr} = \frac{\gamma R}{\kappa(\gamma-1)}\mu \qquad \mathrm{Re} = Lv_0\mu^{-1}\tau_0^{-1}$$

where L is the reference length, v_0 is the reference velocity and τ_0 is the reference specific volume. In system (27) the wave '$\sim$' is dropped. The system of equations (27) is invariant with respect to the transformation

$$\xi' = -\xi \qquad c_1' = -c_1.$$

Therefore, it is enough to study this system for $c_1 \geqslant 0$. If $c_1 \neq 0$, then the system can be solved with respect to the second derivatives of the functions G, V and P. If $c_1 = 0$, then from

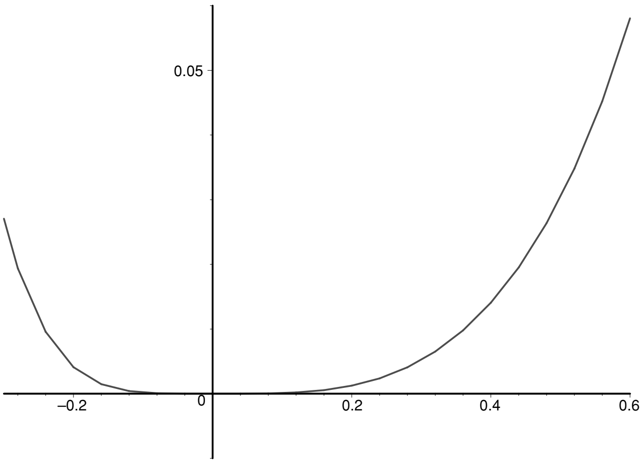

Figure 1. The functions $V(\xi)$ for $c_1 = 0$ and $c_1 = 0.5$.

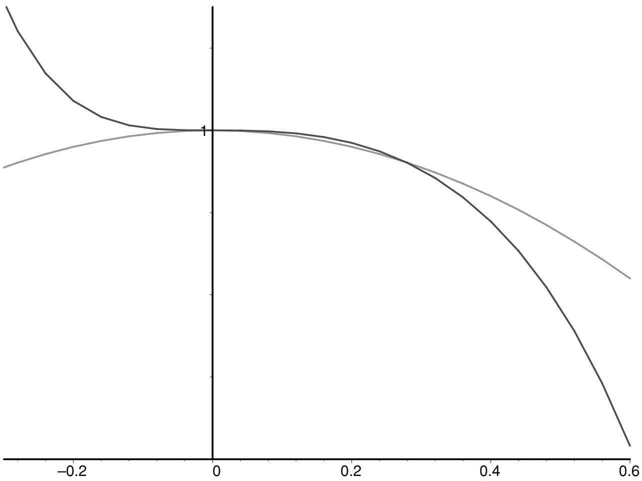

Figure 2. The functions $\tau(\xi)$ for $c_1 = 0$ and $c_1 = 0.5$.

the first equation one can find the first derivative of the function P and the remaining equations can be solved with respect to the second derivatives of the functions V and G. In this case if $V'(0) = G'(0) = P'(0) = 0$, then the functions $V(\xi)$, $G(\xi)$, $P(\xi)$ are symmetric. Note also that if $c_1 = 0$, then there is the particular solution

$$V = 0 \qquad P = C_2 \qquad G = C_3\xi + C_4(\xi^2 - 1).$$

In the figures two solutions with $c_1 = 0$ and $c_1 = 0.5$ are given. The functions for $V(\xi)$ are shown in figure 1. Note that for $c_1 = 0$ the function $V(\xi) = 0$. The functions for $\tau(\xi)$ are shown in figure 2. The function $\tau(\xi)$ for $c_1 = 0$ is symmetric. The functions for $P(\xi)$ are shown in figure 3. The initial values for these solutions are

$$V(0) = 0 \qquad V'(0) = 0 \qquad G(0) = 1.0$$
$$G'(0) = 0 \qquad P(0) = 1.0 \qquad P'(0) = 0$$

and $\mathrm{Pr} = 0.72$, $\mathrm{Re} = 10.0$, $\gamma = 1.4$. Note that increasing Re narrows the domain of validity of the solution.

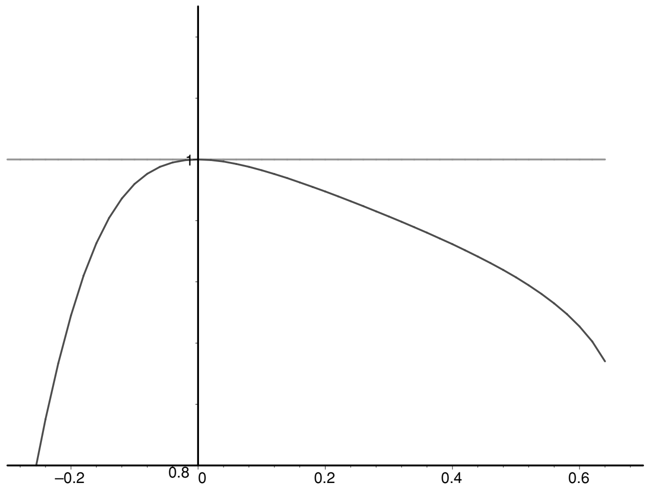

Figure 3. The functions $P(\xi)$ for $c_1 = 0$ and $c_1 = 0.5$.

5.2. *The case* $2\beta + 1 = 0$

If $2\beta + 1 = 0$, then the representation of the invariant solution is

$$u = U(x)\mathrm{e}^{qy} \qquad v = V(x)\mathrm{e}^{qy} \qquad \tau = G(x)\mathrm{e}^{2qy} \qquad p = P(x)$$

where $q = h^{-1}$. Substituting the representation of the invariant solution into the initial system of viscous gas dynamics equations gives the equations

$$\begin{aligned}
&G^{-1}\frac{\mathrm{d}G}{\mathrm{d}x}\left(2\frac{\mathrm{d}U}{\mathrm{d}x}M + \Lambda W\right) - 2\frac{\mathrm{d}P}{\mathrm{d}x}\left(2M'\frac{\mathrm{d}U}{\mathrm{d}x} + \Lambda' W\right) + 2\frac{\mathrm{d}P}{\mathrm{d}x}\sqrt{G}\\
&\qquad - 4\frac{\mathrm{d}^2U}{\mathrm{d}x^2}M - 2\frac{\mathrm{d}W}{\mathrm{d}x}\Lambda + 2G^{-1/2}UW = 0\\
&MG^{-1}\frac{\mathrm{d}G}{\mathrm{d}x}\left(\frac{\mathrm{d}V}{\mathrm{d}x} + qU\right) + 2G^{-1/2}\left(\frac{\mathrm{d}V}{\mathrm{d}x}U + qV^2\right)\\
&\qquad - 2\frac{\mathrm{d}P}{\mathrm{d}x}M'\left(\frac{\mathrm{d}V}{\mathrm{d}x} + qU\right) - 2M\left(q\frac{\mathrm{d}U}{\mathrm{d}x} + \frac{\mathrm{d}^2V}{\mathrm{d}x^2}\right) = 0\\
&\frac{\mathrm{d}G}{\mathrm{d}x}U - G\frac{\mathrm{d}U}{\mathrm{d}x} + qGV = 0\\
&B^{-1}\sqrt{G}\left(PW + \frac{\mathrm{d}P}{\mathrm{d}x}U\right) - \left(\frac{\mathrm{d}U}{\mathrm{d}x}\right)^2(\Lambda + 2M) + \frac{\mathrm{d}U}{\mathrm{d}x}(\sqrt{G}P - 2\Lambda qV) - \left(\frac{\mathrm{d}V}{\mathrm{d}x}\right)^2 M\\
&\qquad - 2\frac{\mathrm{d}V}{\mathrm{d}x}MqU + KR^{-1}\left(\frac{1}{2}\left(\frac{\mathrm{d}G}{\mathrm{d}x}\right)^2 PG^{-1} - \frac{3}{2}\frac{\mathrm{d}G}{\mathrm{d}x}\frac{\mathrm{d}P}{\mathrm{d}x} - \frac{\mathrm{d}^2G}{\mathrm{d}x^2}P\right.\\
&\qquad \left. - \frac{\mathrm{d}^2P}{\mathrm{d}x^2}G - 2Pq^2G\right) - \frac{\mathrm{d}P}{\mathrm{d}x}R^{-1}K'\left(\frac{\mathrm{d}G}{\mathrm{d}x}P - \frac{\mathrm{d}P}{\mathrm{d}x}G\right)\\
&\qquad + q(\sqrt{G}PV - qV^2(\Lambda + 2M) - MqU^2) = 0
\end{aligned}$$

where $W = \frac{\mathrm{d}U}{\mathrm{d}x} + qV$. Note that from the equation of mass conservation, one can find the derivative $\frac{\mathrm{d}U}{\mathrm{d}x}$. The remaining equations are second-order ordinary differential equations with respect to G, V, P.

6. Conclusion

Thermodynamic state equations supplement the basic equations of fluid dynamics and thermodynamics by characterizing the specific fluid of interest. Many special real gas equations exist for specific fluids. The most commonly used thermal equation of state is the thermally perfect gas equation[5], where $p = R\rho T$. The thermally and calorically perfect gas ($\varepsilon = c_v T$) is a polytropic gas.

The general form of the thermal equation of state for real gases is [23]

$$p\tau = RTf(\tau, T)$$

where $f(\tau, T)$ is the gas compressibility factor. The equations of state $(f(\tau, T), \varepsilon(\tau, T))$, coefficients of viscosity and heat conductivity can be obtained from experimental data, derived from the kinetic theory or from an appropriate real gas equation of state. The latter approach is usually used in fluid dynamics. In our study the equations of state are obtained from the requirement of additional symmetry properties. Additional symmetries allow the construction of more exact solutions.

The results obtained in this paper show that the classification of the function $A(p, \tau)$ is similar to the inviscid gas dynamics equations ([22], table 1). There is only one difference: the model 7 ([22], table 1) with the projective generator is absent in our study. The latter is because of (i) the presence of viscosity and (ii) steadiness of studied flows. Classifications of the first $\lambda(p, \tau)$ and second $\mu(p, \tau)$ coefficients of viscosity and the coefficient of heat conductivity $\kappa(p, \tau)$ are related to the classification of the function $A(p, \tau)$. If one uses an additional symmetry for constructing an invariant or a partially invariant solution, then these coefficients must have special representations.

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